Journal of Statistical Physics, Vol. 120, Nos. 3/4, August 2005 (© 2005)
DOI: 10.1007/s10955-005-5959-8

Hysteresis Phenomenon in Deterministic Traffic Flows

Michael Blank[1,*]

Received August 7, 2004; accepted April 29, 2005

We study phase transitions of a system of particles on the one-dimensional integer lattice moving with constant acceleration, with a collision law respecting slower particles. This simple deterministic "particle-hopping" traffic flow model being a straightforward generalization to the well known Nagel–Schreckenberg model covers also a more recent slow-to-start model as a special case. The model has two distinct ergodic (unmixed) phases with two critical values. When traffic density is below the lowest critical value, the steady state of the model corresponds to the "free-flowing" (or "gaseous") phase. When the density exceeds the second critical value the model produces large, persistent, well-defined traffic jams, which correspond to the "jammed" (or "liquid") phase. Between the two critical values each of these phases may take place, which can be interpreted as an "overcooled gas" phase when a small perturbation can change drastically gas into liquid. Mathematical analysis is accomplished in part by the exact derivation of the life-time of individual traffic jams for a given configuration of particles.

KEY WORDS: Dynamical system; traffic flow; phase transition; attractor.

1. INTRODUCTION

By a *traffic flow* we shall mean a collection of particles moving along a straight line according to their velocities, and the law describing how those velocities are changing (i.e., the acceleration or deceleration of particles) is called the *traffic flow model*. For a long time theoretical analysis of traffic flow phenomena has been dominated by hydrodynamic models in analogy to the dynamics of viscous fluid (see, e.g. refs. 9, 12, 17 and

[1]Russian Academy of Sci., Inst. for Information Transm. Problems, and Observatoire de la Cote d'Azur; e-mail: blank@iitp.ru
*This research has been partially supported by Russian Foundation for Fundamental Research and French Ministry of Education grants.

further references therein). The main problem with this approach is that it gives no information about the behavior of individual vehicles. Moreover, the discrete nature of the traffic flow has some features (like traffic jams) which do not have exact counterparts in the hydrodynamic interpretation. Additionally, considered as a kind of viscous fluid, the traffic flow turns out to have very peculiar properties. First, it is very much 'compressible': the distance between two consecutive particles being originally quite far from each other may shrink to zero under the dynamics due to a large jam ahead of them. On the other hand, the opposite property does not hold: the distance between two consecutive particles can be enlarged at most by a constant depending on the parameters of the model but not on time. Mention also that those hydrodynamic models are difficult to treat in computer simulations of large networks, while it is hard to compare parameters of the models with empirical investigations. To overcome these difficulties cellular automata models have been invented in the beginning of 90s (see refs. 4, 6 for reviews).

Consider a system of particles on the integer lattice $\mathbb{Z}^1$ moving with constant acceleration $a \in \mathbb{R}^1_+$ in a discrete time with a collision law respecting slower particles. To make this description precise, we need a few definitions. Each particle is described by the pair (i, x_i), where $i \in \mathbb{Z}^1$ represents the position of the particle and $x_i \in \mathbb{R}^1$ its velocity, which might be both positive and negative. Fix a configuration of particles. For each $i \in \mathbb{Z}^1$ by $i_- \leqslant i$ we denote the site containing the particle from our configuration closest to the site i from the left side, and by $i_+ > i$ the one containing the particle closest to i from the right side (see Fig. 1). Note the asymmetry of the definition: i_- might be equal to i, while i_+ is strictly larger than i.

We shall say that a configuration of particles is *admissible* if for any $i \in \mathbb{Z}^1$ we have:

$$i_+ - i_- > \max\{0,\ x_{i_-},\ -x_{i_+},\ x_{i_-} - x_{i_+}\}, \qquad |x_{i_\pm}| \leqslant v, \tag{1.1}$$

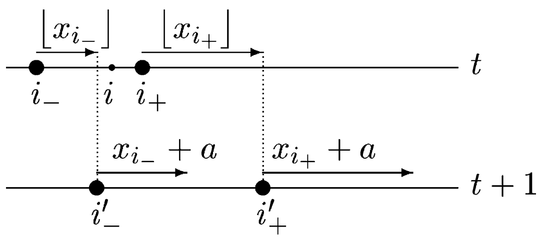

Fig. 1. Freely accelerating dynamics of particles. $i_\pm$ and $i'_\pm$ denote the positions of neighboring particles at moments of time t and $t+1$ and $x_\pm$ – corresponding velocities at time t.

where $v > 0$ is a parameter of the system describing the maximal allowed velocity. In other words, all particles in an admissible configuration can be moved in arbitrary order by the distance equal to the corresponding velocities without collisions.

On admissible configurations the dynamics is defined as follows. First we change simultaneously the coordinates (i, x_i) of all particles according to the rules:

$$i \;\to i + \lfloor x_i \rfloor \tag{1.2}$$

$$x_i \;\to \min\{x_i + a, v\} \tag{1.3}$$

where by $\lfloor\cdot\rfloor$ we denote the integer part of a number, i.e. $\lfloor z\rfloor := \max\{n \in \mathbb{Z}\colon\ n \leqslant z\}$. Denote also by $\lceil\cdot\rceil$ the smallest integer not smaller than the considered number, i.e. $\lceil z\rceil := \min\{n \in \mathbb{Z}\colon\ n \geqslant z\}$.

After this operation for each particle (i, x_i) for which the admissibility condition (1.1) breaks down (which correspond to a 'collision' at the next step of the dynamics) we correct its velocity:

$$x_i \to \min\{x_i,\ \ i_+ - i_- - 1\}. \tag{1.4}$$

Then for all particles where the velocities remain negative we make an additional correction:

$$x_i \to \max\{x_i,\ \ i - (i-1)_- - 1 - \max\{0, x_{(i-1)_-}\}\}. \tag{1.5}$$

It is straightforward to check that after these corrections the configuration of particles becomes admissible.[2]

Figure 2 illustrates the dynamics of our model in the case $a = 1/2$ and $v = 1$. Here and in the sequel we mark the positions of particles in the configuration by their velocities and the positions of holes by dots. The first line in the figure corresponds to a finite segment of length 21 in the initial configuration (i.e. at time 0) and the subsequent lines describe the same segment for the moments of time from 1 to 6. Note that we have chosen the segment such that the particles not shown in the figure do not influence the dynamics of the ones in the segment under consideration during the first six time steps.

Due to the constant acceleration after a finite time (of order v/a) all particles will start moving in the same (positive) direction. Therefore, since

[2] We emphasize that the update rules (1.2)–(1.5) are applied simultaneously to all particles in a given configuration.

1	.	.	.	.	.	-1	.	1	.	0	0	0	.	0	0	.	.	.	.	.
.	1	.	.	.	-a	.	.	.	0	0	0	a	.	0	a	.	.	.	.	.
.	.	1	.	.	0	.	.	.	0	0	0	1	.	0	1	.	.	.	.	.
.	.	.	1	.	a	.	.	.	0	0	a	.	0	a	.	1	.	.	.	.
.	.	.	.	0	1	.	.	.	0	0	1	.	0	1	.	.	1	.	.	.
.	.	.	.	a	.	1	.	.	0	a	.	0	a	.	1	.	.	1	.	.
.	.	.	.	1	.	.	1	.	0	1	.	0	1	.	.	1	.	.	1	.

Fig. 2. An example of the dynamics with $a = \frac{1}{2}$, $v = 1$. The positions of particles are marked by their velocities and the positions of holes by dots.

we shall be interested mainly in the limit behavior of the system as time goes to infinity, to simplify notation we shall assume from now on that velocities of all particles are nonnegative[3] and shall describe the system by a *configuration* $x = \{x_i\}_i \in \mathbb{R}^\infty$, where a nonnegative entry x_i corresponds to a particle at the site i with the velocity $x_i \geqslant 0$, while all other sites (containing 'holes') are set to -1 (having in mind that the 'holes' are moving in the opposite direction to the particles). Under this assumption a configuration x is *admissible* if for any $i \in \mathbb{Z}^1$ we have

$$i_- + x_{i_-} < i_+, \quad x_i \in [0, v] \cup \{-1\}. \tag{1.6}$$

The space of admissible configurations we denote by X. Using this notation the dynamics can be defined as a superposition of two maps $T := \mathcal{A} \circ \sigma$, where the maps $\sigma: X \to \mathbb{R}^\infty$ and $\mathcal{A}: \mathbb{R}^\infty \to X$ are defined as follows:

$$(\sigma x)_i := \begin{cases} x_{i_-} \text{ if } i_- + \lfloor x_{i_-} \rfloor = i \\ -1 \text{ otherwise} \end{cases}, \tag{1.7}$$

$$(\mathcal{A}x)_i := \begin{cases} \min\{x_i + a, \ i_+ - i - 1, \ v\} \text{ if } x_i \geqslant 0 \\ -1 \qquad\qquad\qquad\qquad\quad \text{otherwise} \end{cases}. \tag{1.8}$$

The map σ is a (nonuniform) shift map, describing the simultaneous free shift of all particles in the configuration x by distances equal to their velocities, while the second map $\mathcal{A}$ describes the process of the acceleration/deceleration of particles (see Figs. 1 and 2). Observe that after the application of the map $\mathcal{A}$ any configuration becomes admissible which

[3]Note, however, that due to the argument above all results in the paper remain valid in the initial setting with particle velocities of both signs.

ensures that the map T preserves the space of admissible configurations X.

The dynamics described by the map T for integer values of the parameter $a > 0$ coincides with the deterministic version of the well known Nagel–Schreckenberg model (see e.g., refs. 4 and 6 for reviews). Note also that a version of the Nagel–Schreckenberg model for continuous values of the velocities and spatial coordinates was discussed in ref. 7.

We allow the velocity of a particle to be any real number between 0 and v, but since the particles are moving on the integer lattice the actual shift of a particle (described by the map σ) is an integer, which means that a particle starts moving only when its velocity becomes greater or equal to 1.

One of the most striking observations in the theory of traffic flows is the fact that in a large number of models as time goes to infinity the limit 'average velocity' of the flow depends only on its density and does not depend on the finer characteristics. This dependence is called 'fundamental diagram' in physical literature. Originally this observation has been established numerically for finite lattices with periodic boundary conditions, or in our terms for space periodic configurations.[4] Later this result has been proven rigorously for a broad class of models,[(2,3,15)] assuming that the 'average velocity' is, indeed, the average velocity along a configuration, which we shall call the 'space average velocity'.

Recently a number of nonmarkovian traffic flow models demonstrating the so called 'metastable states' (see, e.g. refs. 1, 8 and 13 and references therein) have been introduced. Numerous numerical simulations of those models show that in distinction to the previous models the 'space average velocity' wildly fluctuates in time. Our own numerical results also show the same phenomenon with fluctuations up to 100% of the 'average' value if the acceleration $a < 1$. To demonstrate this choose a space periodic configuration with the spatial period of length 6 consisting of the periodically repeating pattern [01..1.]. Considering the dynamics with $a = \frac{1}{2}$, $v = 1$ of only the main period of the configuration we get: $01..1. \to a.1..0 \to 1..1.0 \to .1..0a \to \cdots$. As we see starting from the second iterate the 'space average velocity' fluctuates in time periodically. In general the limit as time goes to infinity 'space average velocity' may not exist at all, moreover, both upper and lower limits may fluctuate in time as well. This shows that such approach is not adequate here. On the other hand, the 'space average velocity' tells almost nothing about the movement of individual particles. Therefore, it looks reasonable to consider the averaging in

[4]Indeed, an admissible configuration with a spatial period ℓ remains periodic in space with the same spatial period ℓ which explains the connection to numerical results.

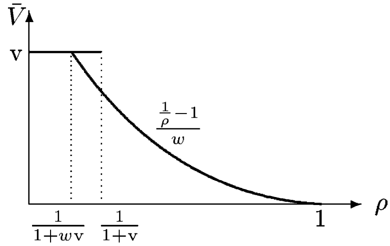

Fig. 3. Fundamental diagram for T: dependence of the limit average velocity $\bar{V}$ on the density of particles ρ. Here $w := \lceil 1/a \rceil$.

time for individual particles instead of the averaging in space. Note that due to the nonergodicity of the process under study these two quantities need not to coincide. On the other hand, it will be shown that the 'time average velocity' behaves much better and the main result of this paper is the proof that the limit 'time average velocity' $\bar{V}(x)$ for an admissible regular[5] configuration $x \in X$ having a density ρ is described by the following multivalued function

$$\mathrm{FD}^{V}_{a,v}(\rho) = \min\{v,\ \frac{1/\rho - 1}{\lceil 1/a \rceil}\} \cup 1_{[(1+\lceil 1/a \rceil v)^{-1},\ (1+v)^{-1}]}(\rho) \cdot v, \qquad (1.9)$$

corresponding to the fundamental diagram shown in Fig. 3. The last term in the above relation describes the upper (unstable) branch of the fundamental diagram. Details related to lower and upper densities/velocities and non-regular configurations will be discussed in Section 3. Let me emphasize that non-regular configurations are not excluded and results related to the fundamental diagram are obtained for all configurations having densities.

Let us mention connections between the model under study and some previously considered cases. In the case when all velocities of particles are positive integers and the acceleration a is an integer as well we immediately recover the deterministic version of the well known Nagel–Schreckenberg model introduced in refs 14. If one assumes that $a \geqslant v$ then our system coincides with the one studied analytically in refs. 2, 3. It is known that in the case $a \geqslant v$ the asymptotic behavior is much simpler, in particular, only free flowing particles or free flowing holes may show up, and, moreover, traffic jams cannot grow in time. There is also another more technical difference due to the fact that in that case the dynamics of holes is completely symmetric to the dynamics of particles. This symmetry was heavily used in refs. 2, 3 for the analysis of the dynamics of high

[5]See definition in Section 3.

density particle configurations. If $a < 1$ there is no independent dynamics of holes as such: to define their dynamics one inevitably needs to take into account the velocities of particles. Therefore, the analysis of a very complicated dynamics of high density particle configurations cannot be reduced to the low density case. Fortunately, as we shall see in the 'jammed' phase the dynamics of individual holes becomes periodic in time (but not in space) after some initial stage. We shall see that if $a < 1$ then when the particle density exceeds some critical value the traffic jams can grow in time and, moreover, an arbitrary small jam (say, consisting of a few particles) may become arbitrary large with time.

It is worth note that when the reciprocal to the acceleration a is an integer and the initial configuration has only integer velocities the model we are discussing coincides with the recent 'slow-to-start' model, which was extensively studied on the numerical level (see e.g. refs. 8, 13, 16 and references therein), but no analytical results have been obtained so far. This model (being non-markovian in the sense that the dynamics depends on the history) turned out to be the first one demonstrating large fluctuations in 'space average' statistics or 'metastable states'.

In this paper, we restrict the analysis to the case of slow particles with $v = 1$, because the analysis of the fast particles with $v > 1$ and/or several lanes uses a rather different mathematical apparatus – substitution dynamics. Moreover, in the general fast particles case (when $v > 1$) additionally to (static) traffic jams that we consider in this paper there are new types of (dynamic) jams when all particles in a jam might have positive velocities and are moving as a whole with the velocity strictly less than v. These new types of jams lead to additional branches in the fundamental diagram and will be discussed elsewhere. Still since some of technical results obtained in the paper remain valid for any $v \in \mathbb{Z}^1_+$ we shall keep the notation v throughout the paper and shall specify explicitly if only the case $v = 1$ is considered.

The paper is organized as follows. In the next section, we introduce basic notions including the notion of the traffic jam and its characteristics such as a basin of attraction and a life-time, and derive results describing their exact values as functions of a given configuration. Using these results in Section 3 we prove the validity of the fundamental diagram and provide its stability analysis in Section 4. An alternative (more intuitive) proof of the validity of the fundamental diagram via space-time averaging is given in Appendix.

2. DYNAMICS OF TRAFFIC JAMS

For a finite segment $x[n,m]$, $n<m\in\mathbb{Z}^1_+\cup\{0\}=:\mathbb{Z}^1_{\geqslant 0}$ of a configuration $x\in X$ we define the *density of particles* $\rho(x[n,m])$ as the number of particles in the segment $N_{\rm p}(x[n,m])$ divided by the length of the segment, i.e.

$$\rho(x[n,m]):=\frac{N_{\rm p}(x[n,m])}{m-n+1}=\frac{1}{m-n+1}\sum_{i=n}^{m}1_{\geqslant 0}(x_i), \tag{2.1}$$

where $1_{\geqslant 0}(\cdot)$ is the indicator function of the set of nonnegative numbers. The generalization of this notion for the entire infinite configuration $x\in X$ leads to the notion of *lower/upper densities*:

$$\rho_{\pm}(x):=\lim_{n\to\infty}\begin{pmatrix}\sup\\ \inf\end{pmatrix}\rho(x[1,n]), \tag{2.2}$$

where (and in the sequel) lim sup corresponds to the index '+' and lim inf to the index '−'. If the lower and upper densities coincide their common value $\rho(\cdot)$ will be called the *density* of the configuration. According to Birkhoff ergodic theorem for any translationally invariant measure on the integer lattice the set of configurations having densities is the set of full measure. In particular, for any space periodic configuration the density is well defined.[6] Probably the analysis of configurations with well defined densities would suffice, but since some configurations with differing lower and upper densities lead to interesting behavior we shall treat them as well.

The reason to fix $m=1$ in the definition above is that for any given integer m the limit points as $n\to\infty$ of the sequence $\{\rho(x[m,n])\}_n$ are the same and do not depend on m. The asymmetry with respect to the left and right 'tails' of the configuration reflects the fact that roles of those 'tails' are rather different in the long term dynamics. In particular, consider two symmetric (to each other) configurations: $y^{(0)}$ having holes at all negative sites and particles at all positive ones, and $y^{(1)}$ having particles at all negative sites and holes at all positive ones. Any particle in the configuration $y^{(0)}$ has zero velocity and will never move; on the other hand, any given particle in $y^{(1)}$ will eventually with time get the largest possible velocity v.

[6]Thus in numerical simulations only configurations with well defined densities can be observed.

Now we are ready to define time and space average velocities. Since the case when there are no particles in the configurations does not make any sense from the dynamical point of view, from now on we shall assume that any admissible configuration contain at least a finite number of particles. Let $L(x, i, t)$ be the distance covered during t time steps by the particle initially (at $t=0$) located at the site i_- of the configuration x. Denote

$$\bar{V}^{\text{time}}(x, i, t) \equiv \bar{V}(x, i, t) := \frac{1}{t} L(x, i, t) \tag{2.3}$$

$$\bar{V}_{\pm}(x, i) := \lim_{t\to\infty} \binom{\sup}{\inf} \bar{V}(x, i, t) \tag{2.4}$$

$$\bar{V}^{\text{space}}(x[-n, m]) := \frac{1}{N_{\mathrm{p}}(x[-n, m])} \sum_{i=-n}^{m} 1_{\geqslant 0}(x_i) \cdot \lfloor x_i \rfloor \tag{2.5}$$

$$\bar{V}_{\pm}^{\text{space}}(x) := \lim_{n,m\to\infty} \binom{\sup}{\inf} \bar{V}^{\text{space}}(x[-n, m]). \tag{2.6}$$

In other words $\bar{V}^{\text{space}}$ describes the average velocity along all particles in a given configuration, while $\bar{V}$ corresponds to the average in time velocity of a given particle. More precisely these quantities describe average *effective* velocities since we take into account only the actual movement of particles and thus drop fractional parts of their velocities.

As we shall show the time average statistics (in distinction to the space average one), does converge to the limit as time goes to infinity and the result does not depend on the initial site i, i.e. $\bar{V}(x, i) \equiv \bar{V}(x)$.

To this end, let us show that the upper and lower densities are invariant with respect to dynamics.

Lemma 2.1. $\rho_{\pm}(Tx) = \rho_{\pm}(x)$ for any configuration $x \in X$.

Proof. For any $n, m \in \mathbb{Z}_{\geqslant 0}$ we have

$$|N_{\mathrm{p}}(x[-n, m]) - N_{\mathrm{p}}((Tx)[-n, m])| \leqslant 2, \tag{2.7}$$

since during one iteration of the map at most one particle can enter the interval of sites from $-n$ to m (from behind) and at most one particle can leave this interval.[7]

[7] If only nonnegative velocities are considered 1 (instead of 2) would suffice in the inequality (2.7), but in order to be valid in the original setting with velocities of both signs we put 2 here.

By the definition of the lower density there is a sequence of integers $n_j \stackrel{j\to\infty}{\longrightarrow} \infty$ such that

$$\frac{N_{\rm p}(x[1,n_j])}{n_j} \stackrel{j\to\infty}{\longrightarrow} \rho_-(x).$$

On the other hand, due to the inequality (2.7) it follows that $\rho_-(x,)$ is a limit point for partial sums for the configuration Tx. Therefore we need to show only that this is, indeed, the lower limit. Assume, on the contrary, that there is another limit point, call it ξ, for the partial sums for Tx such that $\xi < \rho_-(x)$. Doing the same operations with the partial sums for Tx converging to ξ we can show that this value is also a limit point for the partial sums for the configuration x, and, hence, ξ cannot be smaller than $\rho_-(x)$.

The proof for the upper density follows from the similar argument. ∎

Introduce a *weight* function:

$$w(z) := \begin{cases} \lceil (1-z)/a \rceil & \text{if } z \geqslant 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.8}$$

Referring to this function we shall drop the dependence on z if $z=0$ to simplify notation, i.e. $w(0) \equiv w$.

Lemma 2.2. Let $x \in X$ and let the limit $\bar{V}(x,i)$ be well defined. Then for any $j \in \mathbb{Z}^1$ the limit $\bar{V}(x,j)$ is also well defined and coincides with $\bar{V}(x,i)$.

Proof. Observe that for any $x \in X$ and any $i \in \mathbb{Z}^1$ the particles at the sites i_- and i_+ (call them left and right ones) are consequent. After each iteration of the map T the distance between these particles changes by the difference between their velocities, which might take values between 0 and v. Since the left particle can be slowed down only by the right one, we see that for any moment of time t the distance between the particles can be enlarged at most by wv. Indeed, the longest time spent on the same site by the left particle while the right one already started moving cannot exceed w. On the other hand, the maximal distance the right particle can cover during this time is wv. Thus

$$0 \leqslant (i_+ + L(x,i_+,t)) - (i_- + L(x,i_-,t)) \leqslant i_+ - i_- + wv$$

or

$$i_- - i_+ \leqslant L(x,i_+,t) - L(x,i_-,t) \leqslant wv.$$

Dividing by t and using the definition of the time average velocity we get

$$|\bar{V}(x,i,t)-\bar{V}(x,i_+,t)| \leqslant \max\left\{\frac{1}{t}wv, \quad \frac{1}{t}(i_+ - i_-)\right\} \overset{t\to\infty}{\longrightarrow} 0. \tag{2.9}$$

Thus $\bar{V}(x,i)=\bar{V}(x,i_+)$ for any i. Using the same argument for $(i-1)_-$ and i_+ instead of i one extends this result to neighboring particles, and repeating this argument to all particles in the configuration. ∎

This shows that if the limit $\bar{V}(x,i)$ exists for some site i then the time average velocity is well defined, but we still have to check the existence of at least one 'good' site. On the other hand, as we have shown the average velocity do not depend on the left tail of a configuration. Therefore, it is reasonable to introduce for a configuration $x \in X$ the notion of its *class of equivalence* $\hat{x}$ which includes all configurations from X which coincide with x starting from some site, i.e. $x, y \in \hat{x}$ if $\exists k \in \mathbb{Z}^1$: $x_i = y_i \;\; \forall i \geqslant k$.

Clearly in the absence of obstacles all particles are moving freely. Therefore to understand the dynamics we need to study the motion of 'jams' as the only possible obstacles to the free motion of particles. We say that a segment $x[n,m]$ with $n \leqslant m$ corresponds to a *jam* if

$$0 \leqslant x_i < 1 \quad \forall i \in \{n,\dots,m\}, \tag{2.10}$$

$$x_{n-1} < 0, \quad |x_{m+1}| \geqslant 1. \tag{2.11}$$

In other words the jam $x[n,m]$ is a locally maximal collection of consecutive particles having zero velocities with the possible exception of the *leading* one (located at the site m) whose velocity is strictly less than 1. For example, in the configuration

$$\dots(a)\dots(0)\dots1\dots(00)1\dots(000a).1\dots$$

the jams are marked by parentheses.

The number of particles and their positions in a jam may change with time: leading particles are becoming free and some new particles are joining the jam coming from behind. However, only one such change at a time might happen, and, in particular, a jam cannot split into several new jams. Therefore, we can analyze how a given jam changes with time and the main quantity of interest for us here is the minimal number of iterations

$a = \frac{1}{2}$	.	.	[.	1	.	.	0]	.	0	[.	.	.	0	0]
v = 1		.	.	[.	1	.	a]		a	.	[.	.	0	a]
			.	.	[.	0]			1	.	.	[.	0]	
				.	.	[a]				1	.	.	[a]	
					.	1					1	.	1	

Fig. 4. Dynamics of basins of attraction whose boundaries are marked by square brackets.

after which the jam will cease to exist. Denote by $J(t)$ the segment corresponding to the given jam at the moment t (in this notation $J(0)$ is the original jam). Then by the *life-time* of the jam J we shall mean

$$\tau(J) := \sup\{t: \ |J(t)| > 0, \ t > 0\}, \tag{2.12}$$

where $|A|$ is the length of the segment A.

'Attracting' the preceding particles, a jam plays a role similar to an attractor in dynamical systems theory. Therefore it is reasonable to study it in a similar way and to introduce the notion of the *basin of attraction* (notation $\mathrm{BA}(J)$) of the jam $J := x[n,m]$, by which we mean the segment $x[k,m]$ with $k \leqslant n \leqslant m$ of the configuration x containing all particles which will eventually at positive time join the jam or can be stopped by a particle from the jam J during the time $\tau(J)+1$.[8] Examples of basins of attraction and their dynamics are shown in Fig. 4.

Let us introduce the *weight* $W(x[n,m])$ of a segment $x[n,m]$ with $n \leqslant m$ as

$$\begin{aligned} W(x[n,m]) \ &:= w N_{\mathrm{p}}(x[n,m-1]) + w(x_m) \\ &\equiv w \cdot (m-n) \cdot \rho(x[n,m-1]) + w(x_m). \end{aligned} \tag{2.13}$$

The first term in this expression gives the contribution from all particles in the segment except for the leading one, which is described by the second term.

[8]The last assumption excludes the only possibility for a particle to create a new jam when it does not join an existing jam J but is stopped at the time $\tau(J)+1$ by the last particle in it which just get the velocity 1 but did not start moving yet. Consider an example: $.1.a \to ..01$. Here a particle preceding the jam J containing the only particle with velocity a is stopped at the next moment of time (and creates a new jam) when the jam J already ceased to exist.

Theorem 2.1. Let $v=1$, $a \leqslant 1$ and let $J(0) := x[n,m]$ with $n \leqslant m$ be a jam. Then its basin of attraction $\mathrm{BA}(x[n,m])$ is the minimal segment $x[k,m]$, $k \leqslant n \leqslant m$ for which relations

$$N_h(x[k,m]) + 1 = W(x[k,m]), \tag{2.14}$$

$$x_{k-i} < 1 - ia, \quad i = 1, 2 \tag{2.15}$$

hold true,[9] where $N_h(x[k,m])$ is the number of holes in the segment $x[k,m]$. Moreover,

$$\tau(J) = W(\mathrm{BA}(J)), \tag{2.16}$$

i.e. the life-time of the jam is equal to the weight of its basin of attraction, and under the dynamics the BA of a jam is transformed to the BA of the remaining part of this jam.

Relations (2.14), (2.15) give a very simple algorithm to find the left boundary k of a BA: we move to the left until the number of holes in the segment $x[k,m]$ will become equal to the weight of this segment minus one. After this we check the condition (2.15) and continue the procedure if it does not hold or stop otherwise.

Proof. First, let us rewrite the relation (2.14) in a more suitable way. From the definition of the weight function it follows that for $k \leqslant m$ we have

$$N_{\mathrm{p}}(x[k,m]) = \lceil W(x[k,m])/w \rceil. \tag{2.17}$$

On the other hand, we have the trivial identity

$$m - k + 1 = N_{\mathrm{p}}(x[k,m]) + N_h(x[k,m]),$$

which together with (2.17) yields a new relation equivalent to (2.14):

$$m - k + 2 = W(x[k,m]) + \lceil W(x[k,m])/w \rceil. \tag{2.18}$$

[9]When there is no segment satisfying (2.14) and (2.15) we set $k = -\infty$, i.e. the BA coincides with the entire 'left tail'. Note that the relations (2.14), (2.15) give only an implicit information about the BA, which differs significantly from the result in refs. 2, 3 where the much simpler case $a = v$ has been considered.

Consider now in detail the dynamics of a BA of a jam. Two examples are shown in Fig. 4, where $a=1/2$ and we denote as usual holes by dots and mark particles by their velocities. In both examples the jams consist of particles having initially zero velocities, and the right panel demonstrates the reason of the inequalities (2.15), which exclude too slow particles that cannot join the jam. Observe that if one changes the velocity of the first particle in the right panel from 0 to, say a, then the left boundary of the BA goes to the left since the particle immediately preceding the left bracket will be stopped by the last particle in the former jam at the moment of time when it gets velocity 1.

Let us show that a basin of attraction of a jam either consists of a single site, or its first site is occupied by a hole (i.e. $x_k=-1$). To do this, consider a function

$$\phi(\ell):=W(x[m-\ell,m])+\lceil W(x[m-\ell,m])/w\rceil, \quad \ell\in\mathbb{R}^1_+.$$

According to the definition of the weight of a segment (2) we can write the left boundary of the $\mathrm{BA}(x[n+1,m])=x[k,m]$ as $k=m-\tilde{\ell}$, where $\tilde{\ell}$ is the smallest nonnegative solution to the equation

$$\ell+2=\phi(\ell). \tag{2.19}$$

The function $\phi(\ell)$ is piecewise constant with jumps of amplitude $(w+1)$ at integer points corresponding to the lattice sites i where $x_i\geqslant 0$ (i.e. where there is a particle in the configuration x), and $\phi(0)=w(x_m)+1$. Note that at a jump point the value of ϕ corresponds to the upper end of the jump.

There might be two possibilities:

- $x_m\in[1-a,1)$. Then $\phi(0)=w(x_m)+1=(w-1)+1=w$. Hence, the smallest nonnegative solution to Eq. (2.19) occurs at the origin. This case corresponds to the BA consisting of a single site.
- $x_m\in[0,1-a)$. In this case $\phi(0)=w(x_m)+1>(w-1)+1=w$. Thus the starting point $\phi(0)$ of the graph $\phi(\ell)$ is higher than the starting point of the straight line $\ell+w$. Hence the first intersection between these graphs occurs at a horizontal piece of the graph of $\phi(\ell)$ and thus it corresponds to a hole.

It remains to take into account the inequalities (2.15). If at an intersection point they are satisfied, then the result is proven, otherwise, due to the presence of particles in the segment of length 2 immediately preceding to the interval $[m-\ell,m]$, the graph $\phi(\ell)$ makes some positive jumps of amplitude $w+1$ each. Therefore considering the graph after these jumps we are in the same position as in the second case considered above and can apply the same argument about the intersection point. Checking again the inequalities (2.15) and continuing the procedure (if necessary) we are finishing this part of the proof.

Thus if $k<m$ then the first site k of $x[k,m]$ is occupied by a hole (i.e. $x_k<0$), while the last site m is always occupied by a particle with the velocity $x_m<1$ (since $x[n,m]$ is a jam). Therefore after the application of the map T the velocity of the particle at the site m increases by a and either this particle becomes free and leaves the jam (if $x_m+a\geqslant 1$, which yields that the new leading particle of the jam is located at the site $m':=m-1$), or the velocity remains below the threshold 1 and $m':=m$. In both cases the weight of the segment $x[k,m']$ decreases by 1. Hence, the solution to the equation (2.19) with $m=m'$ occurs now at the point $\tilde{\ell}':=\tilde{\ell}+1$ which yields $k':=k+1$, i.e. the left boundary of the BA is shifted by 1 position to the right, and the resulting segment $x[k',m']$ is again the BA of the remaining part $x[n,m']$ of the jam.

Note also that once we have found the solution to the system (2.14), (2.15) then for all subsequent time steps the inequalities (2.15) will be satisfied automatically, since the particles immediately preceding the BA cannot outstrip the left boundary of the BA which is moving with the velocity 1.

Now, since after each iteration of the map T the weight of the BA decreases by 1, we deduce that the original weight of the BA is equal to the life-time of the jam. ▮

Let us give also a short explanation to the "physical meaning" of the formula (2.18). As we just have shown after each iteration the left boundary of the BA increases by 1, while the right boundary decreases by 1 once per w iterations, and thus the "space-time shape" of the BA represents a skewed to the right triangle (see Fig. 4). When eventually the BA size vanishes, the length of the original BA calculated using these two observations gives the relation (2.18).

This result shows that if all jams in the configuration $x\in X$ have finite BAs (or finite life-times, which is equivalent) then eventually with time all particles will become free.

3. VALIDITY OF THE FUNDAMENTAL DIAGRAM

Let us consider the dynamics of jams in more detail.

Lemma 3.1. Let $v = 1,\ a \leqslant 1,\ y \in X$ and let $\rho_+(y) < \gamma_1 := (1 + wv)^{-1}$. Then, there is a configuration $x \in \hat{y}$ in which only finite life-time jams may be present.

Proof. Choose a representative $x \in \hat{y}$ satisfying the assumption that $\rho(x[-n, 0]) \overset{n\to\infty}{\longrightarrow} \rho_+(y)$ and assume contrary to the statement of the Lemma that there exists a jam $J = x[n, m]$ with $n < m < \infty$ having an infinite BA. (We do not exclude the case $n = -\infty$ here.) Then by (2.18) using the same construction as in the proof of Theorem 2.1 for any $k < m$ we get:

$$m - k < W(x[k, m]) + \lceil W(x[k, m])/w \rceil.$$

On the other hand, by (2)

$$W(x[k, m]) \leqslant \rho(x[k, m]) \cdot (m - k + 1) \cdot w.$$

Hence,

$$\rho(x[k, m]) \cdot (m - k + 1) \cdot w + \lceil \rho(x[k, m]) \cdot (m - k + 1) \rceil > m - k.$$

Passing to the limit as $k \to \infty$, we come to a contradiction:

$$\rho_+(x) \geqslant (w + 1)^{-1} \equiv \gamma_1 > \rho_+(x). \quad \blacksquare$$

As we shall see the critical value $\gamma_1 := (wv + 1)^{-1}$ gives the upper bound for the existence of a "stable" free-flowing phase. One might think that under the conditions of Lemma 3.1 all configurations with densities below the critical value γ_1 cannot have infinite life-time jams. To show that this is not the case consider the configuration $y^{(1)}$ having particles only at sites with negative coordinates. Then $\rho(y^{(1)}) = 0$, but the segment $y^{(1)}[-\infty, -2]$ of this configuration corresponds to a jam, and since it has an infinite number of particles, the life-time of this jam is infinite as well.

Lemma 3.2. Let $v = 1,\ a < 1,\ x \in X$ and let in the configuration x there exist a jam $J(t)$ with the infinite BA (i.e., $|J(t)| > 0\ \forall t \geqslant 0$). Then each hole in x, located originally to the right of $J(0)$, starting from the moment of time when this hole meets with the particle that was the leading one in the jam $J(0)$ will start moving periodically by one position to the left exactly once per w time steps.

An immediate extension of this result is the following observation. Let us call the largest moment of time at which a given hole meets a particle the life-time of this hole. Then under the assumption of Lemma 3.2 the motion of each hole having an infinite life-time is eventually periodic with the period w.

Proof. Observe that since there is at least one infinite life-time jam, then each particle belonging to its BA (i.e. located to the left of J) will eventually join the jam J. On the other hand, according to the dynamics the leading particle in the jam J is becoming free exactly once per w time steps. Thus these free particles are passing holes (i.e. exchanging positions with them) exactly once per w time steps (needed for the leading particle in a jam to achieve the velocity 1 and to start moving).[10] ∎

This result provides us with the key observation to the calculation of the time-average velocity in the jammed phase.

Assuming that a configuration $x \in X$ satisfies the assumptions of Lemma 3.2 let us consider in detail the dynamics of a hole located initially at a site j to the right of an infinite life-time jam. According to Lemma 3.2 after some finite transient period this hole will start moving periodically by one position to the left once per w time steps. Denote by ϑ_j the duration of the transient period and by K_j the number of particles which the hole will meet during this time. Then we can characterize the deviation of the movement of the hole from the periodic one by the *defect* of the transient period $D_j := K_j - \vartheta_j / w$.

We shall say that a configuration $x \in X$ is *ultimately jammed* if there is a representative $\hat{x}$ from the same equivalence class in which for any $n \in \mathbb{Z}^1$ there is a jam $J_n(t)$ in the configuration $\hat{x}$ with the infinite BA and $J_n(0) > n$, i.e. at the moment $t = 0$ the jam $J_n(0)$ starts further to the right from the site n. Since the dynamics of a particle do not depend on particles located to the left from it we shall assume (to simplify notation) that in an ultimately jammed configuration all sites with nonpositive numbers are occupied by particles with zero velocities.

Now we are ready to formulate the regularity property mentioned in Introduction. Let in an ultimately jammed configuration x holes with nonnegative positions are located at sites $\{j_k\}$. We shall say that the

[10] Note that even when $v > 1$ and some holes may be trapped inside a jam, the situation remains the same due to a similar argument. Moreover, whence a particle will join some jam (non necessarily with an infinite BA) and will go through it, it will start passing holes exactly once per w time steps.

configuration x is *regular* if the functional

$$\mathrm{Reg}(x) := \limsup_{j_k \to \infty} |D_{j_k}|/j_k \tag{3.1}$$

describing statistics of absolute values of normalized defects vanishes. In other words the regularity means that defects of transient periods can grow only at a sublinear rate. Observe that any spatially periodic ultimately jammed configuration is regular[11] and that the property to be ultimately jammed is preserved under the our equivalence relation.

The following statement describes the limit velocity statistics for high density configurations.

Lemma 3.3. Let $v = 1$, $a < 1$, $x \in X$ and let x be an ultimately jammed configuration.[12] Then

$$\left| \bar{V}_{\pm}(x) - \left(\frac{1}{\rho_{\mp}(x)} - 1 \right) w^{-1} \right| \leqslant \left(\frac{1}{\rho_{\mp}(x)} - 1 \right) \mathrm{Reg}(x). \tag{3.2}$$

Proof. Observe first that we are in a position to apply Lemma 3.2. For any $i \in \mathbb{Z}^1$ the total distance $L(x,i,t)$ covered by the particle initially located at the site i_- in the configuration $x \in X$ during the time $t > 0$ is equal to the number of holes encountered by the particle during this time. For a given $t > 0$ denote by i_t the original position (at moment $t = 0$) in the configuration x of the last hole our particle meets during the time t. Then in the segment $x[i_-, i_t]$ there are exactly $L(x,i,t)$ holes and, hence, its length $i_t - i_-$ can be found from the equation:

$$(i_t - i_-) \cdot (1 - \rho(x[i_-, i_t])) = L(x,i,t),$$

while the number of particles in this segment is equal to

$$N_{\mathrm{p}}(x[i_-, i_t]) = (i_t - i_-) \cdot \rho(x[i_-, i_t]).$$

Thus

$$N_{\mathrm{p}}(x[i_-, i_t]) = L(x,i,t) \cdot \frac{\rho(x[i_-, i_t])}{1 - \rho(x[i_-, i_t])}. \tag{3.3}$$

[11] Clearly this property holds if distances between successive infinite life-time jams are uniformly bounded. See Lemma 3.7 for more general situations.

[12] According to Lemma 3.1 if the density is below γ_1 then all jams have finite BAs. Therefore the assumption about the presence of jams with infinite BAs yields the implicit assumption on the density.

During the time t the hole initially located at the site i_t will meet with all those $N_{\rm p}(x[i_-, i_t])$ particles. Since we are interested in the long term dynamics we may assume that t is so large that there is at least one infinite life time jam between the sites i_- and i_t, which guarantees that $t > \vartheta_{i_t}$. Therefore

$$N_{\rm p}(x[i_-, i_t]) = \frac{t - \vartheta_h(i_t)}{w} + K_{i_t} = \frac{t}{w} + D_{i_t}.$$

Substituting the obtained relation to Eq. 3.3 we get

$$\frac{t}{w} + D_{i_t} = L(x, i, t) \cdot \frac{\rho(x[i_-, i_t])}{1 - \rho(x[i_-, i_t])},$$

which gives

$$\begin{aligned} \bar{V}(x, i, t) &= \frac{1}{t} L(x, i, t) \\ &= \frac{1}{w}\left(\frac{1}{\rho(x[i_-, i_t])} - 1\right) + \left(\frac{1}{\rho(x[i_-, i_t])} - 1\right)\frac{D_{i_t}}{t}. \end{aligned} \tag{3.4}$$

Observe now that $t \geqslant i_t$ and hence

$$|D_{i_t}|/t \leqslant |D_{i_t}|/i_t.$$

Therefore passing to the upper/lower limit as time goes to infinity and taking into account that $i_t \overset{t\to\infty}{\longrightarrow} \infty$ we get the result. ∎

Another more intuitive argument for the proof of this statement based on a space-time averaging will be discussed in the Appendix.

The following result demonstrates the importance of the regularity assumption for the convergence of the limit velocity.

Lemma 3.4. Let x be an ultimately jammed configuration with $\mathrm{Reg}(x) > 0$. Then even if $\rho_-(x) = \rho_+(x) = \rho(x) < 1$ the limit points of the time average velocity may differ from the value $(1/\rho(x) - 1)/w$.

Proof. If the regularity assumption is not satisfied then there exists a sequence of positions of holes $0 < j_1 < j_2 < \cdots$ such that

$$\lim_{k\to\infty} |D_{j_k}|/j_k = \mathrm{Reg}(x) > 0.$$

Assume contrary to our claim that $\bar{V}(x) = (1/\rho(x) - 1)/w$. Then $i_t/t \overset{t\to\infty}{\longrightarrow} \bar{V}(x) > 0$ and therefore the second term in the relation (3) describing fluctuations around the time average velocity does not vanish with time. We came to the contradiction. ∎

Corollary 3.5. Let x be an ultimately jammed configuration with the density $\gamma_1 < \rho(x) < \gamma_2$ and with $\mathrm{Reg}(x) > 0$ and let the lower limit of normalized defects differ from the upper one. Then using the same argument as above one can show that the values $\bar{V}_\pm(x)$ differ as well.

Lemma 3.6. The assumption in Lemma 3.3 that the configuration x is ultimately jammed is a necessary one and it is always satisfied if

$$\rho_-(x) > \gamma_2 := (1+v)^{-1}. \tag{3.5}$$

Proof. If there is only a finite number of jams then particles originating from them might meet holes that were never met by other particles before. Assume now that in the configuration x only finite life-time jams can be present. Since a free particle should have at least v holes immediately ahead of it and since the length of the BA of a jam of n particles exceeds $w(n-1)$, we deduce that in any segment of length ℓ there are at most $\ell/(v+1)$ particles, provided that ℓ is large enough.

It remains to show that under the assumption (3.5) there is an infinite number of infinite life-time jams. Assume that this is not the case, i.e. there exists $n \in \mathbb{Z}^1$ such that in the segment $x[n,\infty]$ only finite life-time jams are present. Then, we can apply the argument in the first part of the proof to this segment to demonstrate that this assumption leads to the inequality $\rho_-(x[n,\infty]) < \gamma_2$. We came to the contradiction since $\rho_-(x[n',\infty]) = \rho_-(x[n,\infty])$ for any $n' < n$. ∎

A more detailed analysis of the regularity functional improves substantially the result of Lemma 3.3. Moreover, the sufficient condition for the regularity property can be formulated in terms of *gaps* $G_k := x[n_k, m_k]$ between successive infinite life-time jams. Define

$$\overline{\mathrm{Reg}}(x) := \limsup_{k\to\infty} |G_k|/m_k. \tag{3.6}$$

Lemma 3.7. Let $v = 1$, $a < 1$, $x \in X$ and let x be an ultimately jammed configuration. Then

(a) $0 \leqslant \overline{\mathrm{Reg}}(x) \leqslant 1/2$,

(b) $\overline{\mathrm{Reg}}(x) \geqslant \mathrm{Reg}(x)$, hence $\overline{\mathrm{Reg}}(x) = 0$ implies $\mathrm{Reg}(x) = 0$,

(c) $\overline{\mathrm{Reg}}(x) = \mathrm{Reg}(x) \equiv 0$ if $\rho_-(x) = \rho_+(x) > \gamma_2$.

Proof. Consider a gap G_k between two successive infinite life-time jams J_k, J_{k+1}. According to its definition the gap may contain only free particles and finite life-time jams together with their basins of attraction (otherwise those jams will have an infinite life-time). Therefore for each particle in the gap there should be at least one hole, which yields the assertion (a).

To prove the assertion (b) observe that for a hole located initially at a site j we have $0 \leqslant K_j \leqslant \vartheta_j$ and thus

$$|D_j| = |K_j - \vartheta_j/w| \leqslant (1 - 1/w)\vartheta_j.$$

Now the statement follows from the observation that for a hole located in the gap G_k the length of the transient period cannot exceed $|G_k| + w$.

Assume now contrary to the last assertion that there is a configuration $x \in X$ with $\rho_-(x) > \gamma_2$ and $\overline{\mathrm{Reg}}(x) > 0$. Then there is a sequence of gaps G_k in x such that

$$\frac{|G_k|}{m_k} \overset{m_k \to \infty}{\longrightarrow} \overline{\mathrm{Reg}}(x) > 0.$$

According to the proof of the assertion (a) the density of particles in any gap G_k cannot exceed $1/2$. Therefore considering the configuration x as a sequence of successive alternative blocks of particles J_k and gaps G_k and taking into account that lengths of jams J_k should go to infinity with k (otherwise $\rho_-(x) \leqslant \gamma_2 = 1/2$ since $|G_k| \overset{k \to \infty}{\longrightarrow} \infty$) we see that either $\overline{\mathrm{Reg}}(x) = 0$ or $\rho_-(x) < \rho_+(x)$ (since x consists of alternative linearly growing blocks of two types J_i with $\rho(J_i) = 1$ and G_i with $\rho(G_i) \leqslant 1/2$). ∎

From these technical statements we derive the main result about the fundamental diagram.

Theorem 3.1. Let $v = 1$, $a \leqslant 1$, $x \in X$. If

(a) $\rho_+(x) < \gamma_1 := (1 + wv)^{-1}$ then $\bar{V}(x) = v$,

(b) $\rho_-(x) > \gamma_1$ and x is not ultimately jammed (hence $\rho_-(x) < \gamma_2$) then $\bar{V}(x) = v$,

(c) $\rho_-(x) > \gamma_1$ and x is ultimately jammed then

$$\left| \bar{V}_\pm(x) - \left(\frac{1}{\rho_\mp(x)} - 1 \right) w^{-1} \right| \leqslant \left(\frac{1}{\rho_\mp(x)} - 1 \right) \mathrm{Reg}(x),$$

(d) $\rho_-(x) = \rho_+(x) = \rho(x) > \gamma_2$ then $\bar{V}(x) = (\frac{1}{\rho(x)} - 1) w^{-1}$.

Proof. By Lemma 3.1 under the condition $\rho_+(x) < \gamma_1$ only jams with finite life-times may be present in x. Consider a partition of $\mathbb{Z}^1$ by nonintersecting finite BAs of jams in the configuration x and their complement. Choose one of those BAs and denote by i the site containing the first particle preceding this BA. Even if originally this particle does not have velocity one it will get this velocity (and will start moving) at most after w iterations. On the other hand, by the definition of the BA this particle (which does not belong to any BA to the right of it) will never join a jam after the first moment of time when $\bar{V}(x,i,t)=1$. Finally, Lemma 2.2 finishes the proof of item (a).

To prove the existence of the upper branch of the fundamental diagram it is enough to show that for any $0<\gamma<\gamma_2 := (1+v)^{-1}$ there are configurations with particle density γ and such that eventually with time all their particles will become free. To demonstrate this, consider a space periodic configuration $x \in X$ with the space period of length 2ℓ in which only even sites $0, 2, \ldots, \lfloor 2\ell\gamma \rfloor$ are occupied by particles with velocity 1, while all other sites are occupied by holes. Clearly, all particles in this configuration are free and will remain free under dynamics. On the other hand, the density of this configuration $\rho(x)=\frac{1}{2\ell}\lfloor 2\ell\gamma \rfloor \stackrel{\ell\to\infty}{\longrightarrow} \gamma$.

The remaining part of Theorem 3.1 describing the lower branch of the fundamental diagram follows immediately from Lemmas 3.3, 3.6, 3.7. ∎

Obtained results give a complete characterization of the time statistics of a configuration x for which the density $\rho(x)$ is well defined or if it is not well defined but either $\rho_+(x)<\gamma_1$ or $\rho_-(x)>\gamma_2$. Let us show that there are other situations with much wilder behavior.

Lemma 3.8. Let $v=1$, $a\leqslant 1$. Then there exists a configuration $y\in X$ such that $\rho_-(y)=0$, $\rho_+(y)=1$ and $\bar{V}_-(y)=0$, $\bar{V}_+(y)=1$.

Proof. We shall construct the configuration y as follows. First we set $y_i := -1 \quad \forall i\leqslant 0$, i.e. we fill in the non positive sites by holes. Then, we shall fill the remaining sites by alternative blocks B_i consisting either only of particles with zero velocities or only of holes. The lengths $\ell_i := |B_i|$ of those blocks we define inductively:

$$\ell_1 := 1, \quad \ell_2 := w^{\ell_1}, \quad \ell_3 := w^{\ell_1+\ell_2}, \ldots, \; \ell_{k+1} := w^{\sum_{i=1}^k \ell_i}, \ldots$$

Denoting $n_k := \sum_{i=1}^k \ell_i$ we get $\ell_{k+1}=w^{n_k}$ and $n_{k+1}=n_k+\ell_{k+1}=n_k+w^{n_k}$. Therefore

$$\rho(y[1,n_{2k+1}]) \geqslant \frac{\ell_{2k+1}}{n_{2k+1}} = \frac{w^{n_k}}{n_k+w^{n_k}} \stackrel{k\to\infty}{\longrightarrow} 1.$$

Using a similar argument, but counting holes instead of particles we get:

$$1-\rho(y[1,n_{2k}]) \geqslant \frac{\ell_{2k}}{n_{2k}} \overset{k\to\infty}{\longrightarrow} 1.$$

Thus, $\rho(y[1,n_{2k}]) \overset{k\to\infty}{\longrightarrow} 0$. This proves the claim about the lower and upper densities.

Observe now that $\forall k \geqslant 0$ the block B_{k+1} corresponds to a jam of length ℓ_{k+1} and that all particles to the left of this jam lie in the segment $x[1,n_k]$, whose length is equal to $\log_w(\ell_{k+1})$. Therefore by Theorem 2.1 all these particles belong to the basin of attraction of this jam and hence will join it with time. On the other hand, since there are no particles at negative sites, this shows also that life-times of all jams in the configuration y are finite. Note, however, that the life-time of the k-th jam is of order $\exp(k)$.

Choose any particle of this configuration and consider how its velocity changes in time. First from $t=0$ to $t=t_1 \geqslant 0$ the particle might stay in a jam and have zero velocity. Then it is becoming the leading one and during w time steps preserves its position but accelerates until it get velocity 1. After that the particle starts moving freely with the velocity 1 until it will catch up with the next jam. Then, it will again stay in a jam having the zero velocity, etc. Due to the calculations above the duration of the alternative periods of staying in a jam and free moving (interrupted by short periods of acceleration of length w) are growing exponentially. From this we get immediately the statement about the lower and upper time average velocities. ∎

4. STABILITY/INSTABILITY OF THE FUNDAMENTAL DIAGRAM

Qualitatively the main difference between the configurations belonging to the upper and lower branches of the fundamental diagram is related to the property to be ultimately jammed or not. According to Theorem 3.1, we need to study the stability of this property only for configurations with densities in the region (γ_1,γ_2) where the two branches of the diagram coexist. The following result demonstrate instability of the upper branch of the fundamental diagram, while the stability of the lower branch is discussed in Theorem 4.2.

Theorem 4.1. Let $v=1$, $a<1$, $x\in X$ and let the density $\rho(x)$ be well defined and $\rho(x)>\gamma_1$ while $\bar{V}(x)=1$ (i.e. the configuration x belongs to the upper branch of the fundamental diagram). Then, there exists an

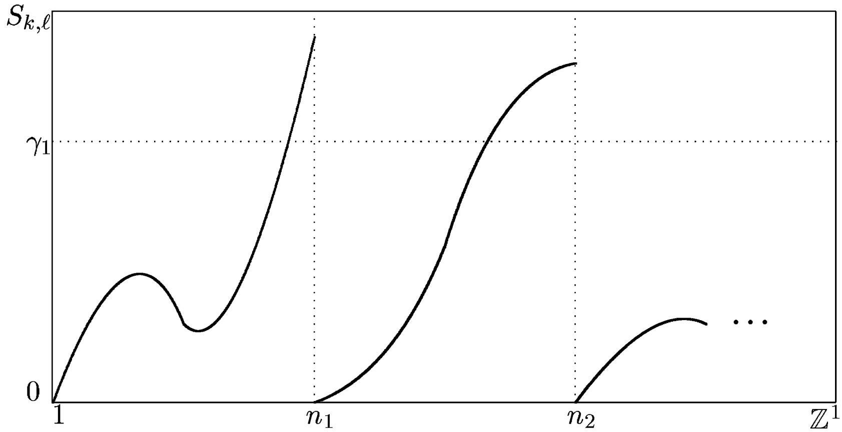

Fig. 5. Averages along a configuration.

ultimately jammed configuration $y \in X$ which differs from x on a set S having zero density.

Proof. We shall construct the perturbed configuration y as follows. Set $\ell := 0$ and for for all $k \in \mathbb{Z}^1_+$ for which $x_k \geqslant 0$ consider a sequence of numbers $S_{k,\ell} := \rho(x[\ell+1, k])$. We denote by n_1 the value of k which gives the first local maximum of $S_{k,\ell}$ for which $S_{k,\ell} > \gamma_1$. Then we set $\ell := n_1$ and continue the procedure to find the number n_2 giving the local maximum exceeding the value γ_1, etc. (see Fig. 5). Eventually we shall have a monotonically growing sequence $n_i \stackrel{i\to\infty}{\longrightarrow} \infty$, from which we can further choose a subsequence $\tilde{n}_i$ satisfying the assumption that $|\tilde{n}_i - \tilde{n}_{i+1}| \geqslant 2^i$. Observe that the set of integers $\{\tilde{n}_i\}_i$ has zero density.

Let $y_j := 0$ for all $j <= 0$ and for positive j we set $y_j \equiv x_j$ except for the sites $\tilde{n}_i$ where we set $y_{\tilde{n}_i} := 0$, i.e. we have changed the velocities of the particles at the sites $\tilde{n}_i$ to 0 for all i. Since $\tilde{n}_i$ is the point of a local maximum, the site $\tilde{n}_i$ is the leading point of some jam in y. Our claim is that the BA of this jam covers the entire segment $[\tilde{n}_{i-1}+1, \tilde{n}_i]$. Assume that this is not the case, i.e. there exists $N < \tilde{n}_i$ such that the BA corresponds to the segment $y[\tilde{n}_i - N + 1, \tilde{n}_i]$ and denote by M the number of particles in this segment. Due to the local monotonicity of the sequence $S_{k,\ell}$ in the segment preceding the point $\tilde{n}_i$ the strict inequality $\gamma_1 < M/N$ holds true. On the other hand, by Theorem 2.1 we have

$$N = W(y[\tilde{n}_i - N + 1, \tilde{n}_i]) + \lfloor W(y[\tilde{n}_i - N + 1, \tilde{n}_i])/w \rfloor = wM + M.$$

Thus $M/N = \frac{1}{1+w} = \gamma_1$, which contradicts to our assumption.

Now, since the density of differing sites is equal to zero (due to the choice of $\tilde{n}_i$) while the configuration y satisfies the assumptions of Lemma 3.3 we get the result. ▮

Clearly, changing back the zero density set of sites in the ultimately jammed configuration y constructed in the proof above we get the configuration x in which all particles will eventually move freely. Therefore, one might expect that all ultimately jammed configurations with densities in the interval (γ_1, γ_2) are also unstable. The following statement shows that this is not the case.

Theorem 4.2. Let $v=1$, $a<1$. There exists a regular configuration $x \in X$ with $\gamma_1 < \rho(x) < \gamma_2$ and constants $0 < A < B < \infty$ such that if a configuration $y \in X$ differs from x at most at A sites in any segment of length B then y is also regular. In other words, there is an open neighborhood of the configuration x consisting only of configurations corresponding to the lower branch of the fundamental diagram.

Proof. Choose positive integers N, M large enough such that $\gamma_1 < M/(N+M) < \gamma_2$ and consider a spatially periodic configuration x whose main period $x[1, N+M]$ consists of N consecutive holes and M consecutive particles having zero velocities. Then, the density $\rho = \frac{M}{N+M}$ of this configuration is well defined and this configuration evidently satisfies the conditions of Lemma 3.3.

Our aim is to show that for a small enough (but still nondegenerate) perturbation the property to have a countable number of infinite life-time jams arbitrary far to the right is preserved. Moreover we shall see that there exists a finite N_1 such that any segment of length N_1 in the perturbed configuration y intersects with some infinite life-time jam. Clearly additional particles in y (compared to x) should not worry us and it is enough to consider only the case when under the perturbation some particles are removed from the configuration x or their velocities are changed.

Choose $n = k(N+M)$ with $k \in \mathbb{Z}_+$ large enough and assume on the contrary that all jams in $y[1, n]$ have finite life-times and their BAs lie completely in this segment (i.e. they do not include the site 0). Denote by m_0 the number of free particles in the segment $y[1, n]$ and by m_1 the number of particles belonging to jams in this segment.

By Theorem 2.1, we know that if I is a finite segment corresponding to a basin of attraction then $W(I) > wN_{\mathrm{p}}(I)$ and $N_{\mathrm{h}}(I) > W(I)$. Thus

$$|I| = N_{\mathrm{p}}(I) + N_{\mathrm{h}}(I) > (w+1)N_{\mathrm{p}}(I).$$

Taking into account that each free particle is followed by a hole we conclude that

$$2m_0 + (w+1)m_1 \leqslant n.$$

On the other hand,

$$m_0 + m_1 = N_{\mathrm{p}}(y[1,n]) \leqslant n\rho.$$

Thus

$$\frac{m_0}{n} \geqslant \frac{(w+1)\rho - 1}{w-1} > 0.$$

Observe now that in order to get a free particle in the configuration y we need to change at least two sites in the configuration x. From this one immediately can find the constants A, B setting

$$A := \left\lfloor \frac{(w+1)\rho - 1}{w-1} \cdot B \right\rfloor$$

and choosing $B := 2k(N+M)$ to be so large that $A \geqslant 1$. ▮

Using the same argument, one can prove stability for a regular configuration for which instead of space periodicity one assumes that the gaps between infinite life-time jams are uniformly bounded.

Let us show now that a single site perturbation of a configuration consisting only of free particles can create a jam whose length grows linearly with time. For $v = 1$, $a = 1/2$ consider a configuration x having free particles (with velocity 1) at all even sites and holes at all others. We perturb this configuration only at the origin setting the velocity at site 0 to zero and thus creating a jam of length 1. According to Theorem 2.1, the BA of this jam is infinite. On the other hand, after each iteration a new particle joins this jam from the left while only once per two iterations the leading particle leaves the jam, which proves its linear growth.

A partial result in the direction of a measure-theoretic interpretation of the notion of regularity for configurations in the density region (γ_1, γ_2) gives the following Lemma.

Lemma 4.1. Let a configuration x has a density $0 < \rho(x) < 1$. Decompose x into alternative blocks of particles A_i and holes B_i: $x = \dots A_1 B_1 A_2 B_2 \dots$. Then

$$\frac{|A_n| + |B_n|}{\sum_{i=1}^{n}(|A_i| + |B_i|)} \stackrel{n\to\infty}{\longrightarrow} 0.$$

In other words, the lengths of blocks can grow only at a sublinear speed.

Proof. Let us prove that lengths of blocks of holes cannot grow at a linear speed. Assume contrary to our claim that there exists a constant $0 < \zeta < 1$ and a sequence of integers $0 < n_1 < m_1 < n_2 < m_2 < \cdots$ with $n_i \leqslant \zeta m_i$ for all i such that all sites in the segments $x[n_i, m_i]$ are filled by holes while the segments $x[m_i, n_{i+1}]$ are filled by particles. Since the density of the configuration x is well defined we have:

$$\rho(x[1,k]) \stackrel{k\to\infty}{\longrightarrow} \rho(x).$$

On the other hand,

$$\rho(x[1,m_i]) = \frac{n_i \rho(x[1,n_i])}{m_i} \leqslant \zeta\rho(x[1,n_i]) \stackrel{i\to\infty}{\longrightarrow} \zeta\rho(x),$$

which contradicts to the assumption that the density of x is well defined. A similar argument applies to blocks of particles as well. ∎

Recalling the connection between the functionals Reg and $\overline{\text{Reg}}$ we see that the set of regular configurations do not differ much from the set of all configurations having densities even in the "hysteresis" region and we expect that for any nontrivial translationally invariant measure on X the set of regular configurations is a set of full measure.

5. CONCLUDING REMARKS

A deterministic generalization of the Nagel–Schreckenberg traffic flow model (as well as of the slow-to-start model) with the real valued acceleration has been proposed and studied analytically. It has been shown that macroscopic physical properties of the model under study such as the time average velocity of particles in the flow depend crucially on the density of particles in the flow and these results are described in terms of the corresponding fundamental diagrams. Moreover, we have shown that instead of space (or space-time) average velocities used earlier in traffic models the time average velocity can be considered. It is proven that this quantity characterizing the behavior of an individual particle coincides for all particles in the flow and in distinction to the space average velocity (which wildly fluctuates in time) leads to a proper statistical description of the dynamics.

One of the problems for future analysis is the question what happens when the initial configuration is random or the dynamics is weakly perturbed in random or deterministic sense (e.g., random traffic lights are taken into account). The main idea in the paper is that without perturbations one can predict the asymptotic behavior of an individual configuration (and even of an individual particle in it). Clearly integrating the results according to some initial distribution one gets predictions for a family of random initial configurations. One would expect that such questions might be of interest only in the 'hysteresis' region where both branches of the fundamental diagram are present. Even there according to Theorem 4.1 the influence of the upper branch should be negligible. Another point is that it might be possible that under arbitrary small perturbations the upper branch of the fundamental diagram will disappear[13] and that the non-regularity of initial configurations should not matter in the random setting. However to check these predictions one needs to study a probabilistic version of the model.

Finally let us mention that it is important to distinguish between peculiarities related to a trivial divergence of various series due to the fact that we deal with the infinite phase space (infinite $\mathbb{Z}^1$ lattice) and true finite-size phenomena taking place even in the case of spatially periodic configurations (or a finite lattice). The latter issue seems more important and our result proves that the hysteresis phenomenon exists even in this case.

APPENDIX: PROOF OF LEMMA 3.3 VIA SPACE-TIME AVERAGING

Lemma 3.3 can be proven also in a more intuitive way using a kind of a space-time averaging. Since this approach explains the connection between the time and space averaging we shall discuss it additionally to the formal proof given in Section 3.

Observe that by the definition of the map T in the case $v = 1$ we explicitly have the mass conservation law, i.e. after each move one

[13] From ref. 10, it follows that if $v=1$, $a=1/2$ and random perturbations are applied only to jams but not to free particles the upper branch exists at least for spatially periodic configurations.

particle exchanges its position with one hole. Therefore, we can give a kind of 'physical derivation' of the relation (3.2). Assume that the (time) average velocity $\bar{V}(x)$ is well defined and choose a segment of the configuration x of length ℓ, provided $\ell \gg 1$. Then the number of particles in this segment is equal to $N_{\rm p} := \ell(\rho(x) + o(1/\ell))$, while the number of holes is equal to $N_{\rm h} := \ell(1-\rho(x)+o(1/\ell))$. We are in a position to apply Lemma 3.2, hence the holes are moving with the average velocity $1/w$ to the left. Thus in total $N_{\rm p}$ particles will move to the distance $tN_{\rm p}\bar{V}(x)$ during the time t, while in total $N_{\rm h}$ holes will move to the distance $tN_{\rm h}/w$ during this time. Due to the mass conservation these quantities coincide and thus

$$t\ell(\rho(x)+o(1/\ell))\bar{V}(x) = t\ell(1-\rho(x)+o(1/\ell))/w.$$

Dividing by t and ℓ and passing to the limit as $\ell \to \infty$, we get

$$\bar{V}(x) = \frac{1-\rho(x)}{w\rho(x)}.$$

To make this argument rigorous, we need some additional work to be done. We start with the space-periodic case and restrict ourselves to just one period. For any given time interval the total shift of all particles in the 'spatial period' is well defined. Dividing this total shift by the time and by the number of particles in the 'spatial period' and passing to the limit as time goes to infinity (which exists due to the behavior of holes) we get the 'space-time' average velocity. On the other hand, since $\bar{V}(x,i)$ does not depend on i we deduce that our 'space-time' average is, in fact, just $\bar{V}(x)$. Let us prove the last statement. Denote the period length by ℓ. For any moment of time t we have the following relation:

$$\sum_{i=1}^{\ell} 1_{\geqslant 0}(x_i) \cdot L(x,i,t) = \ell(1-\rho(x)) \cdot \frac{t}{w} + R(t),$$

Here (due to the usage of the indicator function) the summation is taken over all particles in the 'spatial period', the remaining term $R(t)$ cannot exceed ℓ/w on absolute value, and the value $\ell(1-\rho(x))$ is equal to the number of holes in the spatial period. Dividing by t, we get

$$\sum_{i=1}^{\ell} 1_{\geqslant 0}(x_i) \cdot \bar{V}(x,i,t) = \ell(1-\rho(x)) \cdot \frac{1}{w} + O(1/t), \tag{5.1}$$

where $O(1/t)$ means a term of order $1/t$ as $t \to \infty$.

On the other hand, by the inequality (2.9) for any two particles initially located at sites i, j of the 'spatial period' and any moment of time t we have:

$$|\bar{V}(x,i,t) - \bar{V}(x,j,t)| \leqslant \ell \max\left\{\frac{1}{t} wv, \quad \frac{1}{t}|i-j|\right\} \leqslant \ell^2/t,$$

provided that ℓ is large enough. Thus

$$\left| \ell\rho(x) \cdot \bar{V}(x, \ell_-, t) - \sum_{i=1}^{\ell} 1_{\geqslant 0}(x_i) \cdot \bar{V}(x,i,t) \right| \leqslant \ell^3/t,$$

where the value $\ell\rho(x)$ is equal to the number of particles in the 'spatial period'.

Therefore passing to the limit as $t \to \infty$ in the relation (5.1) and using the inequality above, we get

$$\lim_{t\to\infty} \bar{V}(x, \ell_-, t) = \frac{\ell(1-\rho(x))}{\ell\rho(x)} \cdot \frac{1}{w} = \left(\frac{1}{\rho(x)} - 1\right) w^{-1},$$

which finishes the proof in the space-periodic case.

In the general non space-periodic case one also can follow a similar argument. According to our assumption in the configuration x there are finite jams $J_i(t)$ with infinite life-times, denoted by rectangles in Fig. 6. Here the parameter t indicates the moment of time. The segment which starts immediately after the site occupied by the leading particle of the jam $J_i(t)$ and finishes at the site occupied by the leading particle of the jam J_{i+1}, we denote by $I_i(t)$. Then one can proceed as follows.

(a) The number of holes in $I_i(t)$ is invariant with respect to the dynamics. Indeed, if a hole was located at time t to the right of a jam $J(t)$, then for any moment of time $t' > t$ when $J(t')$ still exists this hole should be still to the right of $J(t')$. The reason is that the right boundary of a jam changes only when its leading particle becomes free. However, the new leading particle of the jam still remains to the left of our hole.

(b) The number of particles in $I_i(t)$ can change due to dynamics at most by 1 in both sides. Indeed, the leading particle of each the jams $J_i(t)$ is becoming free periodically with the period w,

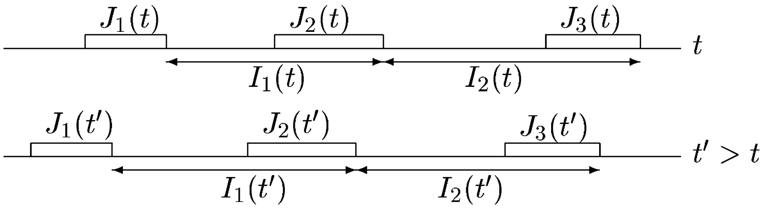

Fig. 6. Dynamics of traffic jams at moments of time t and $t' > t$.

which is the only way how the number of particles in $I_i(t)$ can change.

According to these properties one can think about the holes inside of each segment $I_i(t)$ as a kind of a pump which is pushing the particles through itself with the constant 'productivity' equal to

$$\frac{N_{\mathrm{h}}(I_i(t))}{N_{\mathrm{p}}(I_i(t))} \cdot \frac{1}{w} \ ,$$

which leads to the desired result.

ACKNOWLEDGMENT

The author is grateful to anonymous referees for very helpful comments.

REFERENCES

1. R. Barlovic, T. Huisinga, A. Schadschneider, and M. Schreckenberg, Open boundaries in a cellular automaton model for traffic flow with metastable states, *Phys. Rev. E* **66**, 046113:11 (2002).
2. M. Blank, Dynamics of traffic jams: order and chaos, *Moscow Math. J.* **1**(1):1–26 (2001).
3. M. Blank, Ergodic properties of a simple deterministic traffic flow model, *J. Stat. Phys.* **111**(3–4):903–930 (2003).
4. D. Chowdhury, L. Santen, and A. Schadschneider, Statistical physics of vehicular traffic and some related systems, *Physics Rep.* **329**:199–329 (2000).
5. H. Fuks, Exact results for deterministic cellular automata traffic models, *Phys. Rev. E* **60**:197–202 (1999).
6. L. Gray, and D. Griffeath, The ergodic theory of traffic jams, *J. Stat. Phys.* **105**(3/4):413–452 (2001).
7. S. Krauss, P. Wagner, and C. Gawron, Continuous limit of the Nagel-Schreckenberg model, *Phys. Rev. E* **54**:3707–3712 (1996).
8. S. Krauss, P. Wagner, and C. Gawron, Metastable states in a microscopic model of traffic flow, *Phys. Rev. E* **55**:5597–5602 (1997).

9. R. LeVecque, *Numerical Methods for Conservation Laws*, Lectures in Mathematics ETH Zurich, Birkhauser, Basel, 1990.
10. E. Levine, G. Ziv, L. Gray, and D. Mukamel, Phase transitions in traffic models, *J. Stat. Phys.* **117**(5–6):819–830 (2004).
11. T. M. Liggett, *Stochastic interacting systems: contact, voter, and exclusion processes*, Springer-Verlag, NY, 1999.
12. M. J. Lighthill, G. B. Whitham, On kinematic waves: II. A theory of traffic flow on long crowded roads, *Proc. of R Soc. A* **229**:317–345 (1955).
13. K. Nishinari, M. Fukui, and A. Schadschneider, A stochastic cellular automaton model for traffic flow with multiple metastable states, *J. Phys. A* **37**:3101–3110 (2004).
14. K. Nagel and M. Schreckenberg, A cellular automaton model for freeway traffic, *J. Physique I* **2**:2221–2229 (1992).
15. K. Nishinari, and D. Takahashi, Analytical properties of ultradiscrete Burgers equation and rule-184 cellular automaton, *J. Phys. A* **31**(24):5439–5450 (1998).
16. A. Schadschneider, and M. Schreckenberg, Traffic flow models with 'slow-to-start' rules, *Ann. Physik* **6**:541 (1997).
17. H. M. Zhang, A mathematical theory of traffic hysteresis, *Transport. Res. B* **33**:1–23 (1999).